Physics 81 – Formula Sheet – Learning Assessment 1

Coulomb's Law, Potential Energy, and Electric Fields

For two charges q_1 and q_2, the force on q_1 due to q_2 is given by

$$\vec{F}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r_{12}}$$

Where $\vec{r}_{12}$ is the vector from q_2 to q_1, and $k = 1/4\pi\epsilon_0$ is Coulomb's constant and ϵ_0 is the vacuum permitivity constant.

Computing the work to assemble some configuration of charges by bringing them together from infinitely far away, the energy of a configuration of charges is

$$U = \frac{1}{2}\sum_{j=1}^{N}\sum_{k\neq j}\frac{1}{4\pi\epsilon_0}\frac{q_j q_k}{r_{jk}}$$

The electric field at a point $\vec{r}$ from a charge distribution ρ is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')\,dV'}{s^2}\hat{s} \qquad\qquad \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{s_i^2}\hat{s}_i,$$

where $\vec{s} = \vec{r} - \vec{r}'$ is the vector pointing from a small charge element $dq = \rho(\vec{r}')dV'$ to the point $\vec{r}$ in space (with magnitude s), and where the integral is over $\vec{r}'$, the coordinates of the charge distribution.

The force on a test charge q due to this electric field is given by: $\vec{F} = q\vec{E}$

Charges, charge densities, electric fields and forces exhibit the principle of **superposition** wherein the total charge, or charge density, or electric field a system is the sum of of the individual charges, charge densities, or electric fields, respectively.

An electric field has an energy density: $u = \epsilon_0 E^2/2$ which can be integrated to find the total energy in a region of space V

$$U = \frac{\epsilon_0}{2}\int E^2\,dV$$

Gauss's Law

Gauss's Law relates electric flux through a surface to the charge enclosed by the surface,

$$\Phi = \int_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int\rho\,dV = \frac{1}{\epsilon_0}\sum_i q_i$$

This relation becomes very useful when we can appeal to symmetry and choose a surface where the dot product $\vec{E}\cdot d\vec{a}$ is easy to evaluate.

Some examples of symmetric configurations include a sphere of total charge Q, a line of uniform charge density λ, and a sheet of uniform charge density σ,

$$\vec{E}_{\text{sphere}} = \frac{kQ}{r^2}\hat{r} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \qquad \vec{E}_{\text{line}} = \frac{2k\lambda}{r}\hat{r} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \qquad \vec{E}_{\text{sheet}} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

Where r is the distance radially away from the center of the sphere or the line charge, and $\hat{n}$ is the sheet's surface normal vector.

For a surface charge density σ, the change in the perpendicular component of the electric field across the surface is given by: $\Delta E_\perp = \sigma/\epsilon_0$

Electric Potential

For an electrostatic field $\vec{E}$, the line integral of the field between two points P_1 and P_2 is path-independent (because the field is conservative), allowing us to define an electric potential difference between these two points

$$\Delta\phi = \phi_2 - \phi_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

A test charge q would have a potential energy difference between these points given by: $\Delta U = q\Delta\phi$

If a charge distribution ρ is finite, we can let $\phi(r \to \infty) \to 0$ and define the electric potential in a region of space as

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{s}\, dV' \qquad\qquad \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{s_i}$$

where $\vec{s} = \vec{r} - \vec{r}\,'$ is the vector pointing from a small charge element $dq = \rho(\vec{r}\,')dV'$ to the point $\vec{r}$ in space (with magnitude s), and where the integral is over $\vec{r}\,'$, the coordinates of the charge distribution.

Vector Calculus

We introduced three vector operators in Cartesian coordinates: the gradient of a scalar field, the divergence of a vector field, and the curl of a vector field

Gradient: $\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$

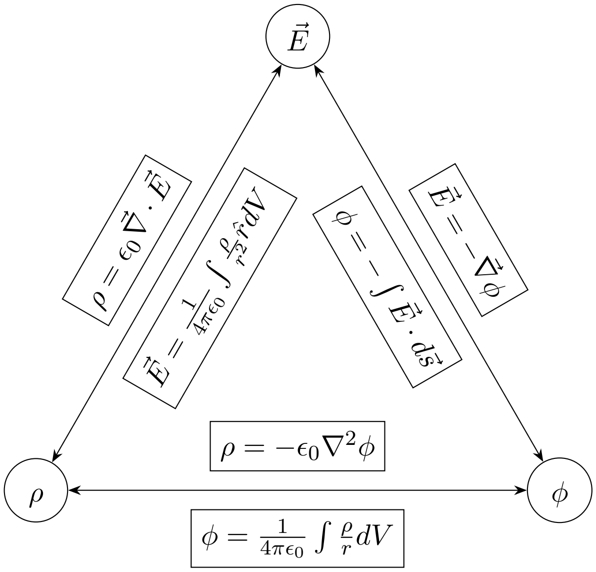
Divergence: $\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

Curl: $\vec{\nabla} \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{k}$

The negative gradient of electric potential is equal to the electric field: $\vec{E} = -\vec{\nabla}\phi$

The divergence of the electric field is equal to the local charge density: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

The curl of a real electrostatic field is always zero: $\vec{\nabla} \times \vec{E} = 0$



Poisson's Equation

If we combine two of the previous operators, applying the divergence to the gradient, we define the Laplacian of a scalar function as

$$\vec{\nabla} \cdot \vec{\nabla}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

We can combine some of the previous relations to derive Poisson's Equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \qquad \text{in the absence of charge:} \quad \nabla^2\phi = 0$$

Conductors

A conductor is a type of material characterized by an abundance of charge carriers (electrons) that are free to move within the conductor. As a result, conductors in equilibrium have these properties

- The electric field $\vec{E}$ inside a conductor is always zero
- The electric potential is constant everywhere inside and on the surface of a conductor
- The electric field directly outside a conductor is perpendicular to the surface of the conductor
- The net charge inside of a conductor is zero, so if there is a charge, it must be contained on the surface of the conductor

If a charge q is contained within a conducting shell, a total charge $-q$ is induced as a surface charge density on the inner surface of the shell such that there is no electric field inside the conductor itself.

The Uniqueness Theorem and Method of Images

The Uniqueness Theorem: If a potential ϕ satisfies Poisson's equation and some boundary conditions on conductors with fixed potential, then that is the only such ϕ in the space between the conductors.

A consequence of the Uniqueness Theorem: in a configuration of charges q_i and conductors, a conducting surface bounding some region can be replaced by an additional system of "image charges" q_j outside that region (thus not altering Poisson's equation in the bound region), such that the total configuration of charges produces the same constant potential at the surface of the conductor.

- The canonical example of the method of images is a charge q a distance d above an infinite conducting plane. The plane can be replaced by a single "image charge" $-q$ a distance d below the conducting plane, which together with the original charge q produces a constant potential at the former location of the conducting plane

Capacitors

A conductor with total charge Q and potential ϕ has a capacitance defined as $C = Q/\phi$. For example, a sphere of radius R in free-space has capacitance

$$C_{\text{sphere}} = 4\pi\epsilon_0 R$$

Two conductors with charge $+Q$ and $-Q$ and potential difference $\Delta\phi$ have a mutual capacitance of $C = |Q|/|\Delta\phi|$. Two parallel conducting plates of area A separated by a distance d have capacitance

$$C_{\text{plates}} = \frac{\epsilon_0 A}{d}$$

The energy stored in a capacitor is given by

$$U = \frac{1}{2}C\phi^2 = \frac{Q^2}{2C} = \frac{1}{2}Q\phi$$